

# Construction and Uniqueness for reflected BSDE under linear increasing condition

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**Abstract.** In this paper, we study the uniqueness of the solution of reflected BSDE with one or two barriers, under continuous and linear increasing condition of generator  $g$ . Before that we study the construction of solution of reflected BSDE with one or two barriers.

## 1 Introduction

El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced in 1997 the notion of reflected BSDE (RBSDE in short) on one lower barrier [1]: the solution is forced to remain above a continuous process, which is considered as the lower barrier. More precisely, a solution for such equation associated to a coefficient  $f$ , a terminal value  $\xi$ , a continuous barrier  $L$ , is a triple  $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$  of adapted processes valued on  $\mathbb{R}^{1+d+1}$ , which satisfies a square integrability condition,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, 0 \leq t \leq T, \text{ a.s.},$$

and  $Y_t \geq L_t$ ,  $0 \leq t \leq T$ , a.s.. Furthermore, the process  $(K_t)_{0 \leq t \leq T}$  is non decreasing, continuous, and the role of  $K_t$  is to push upward the state process in a minimal way, to keep it above  $L$ . In this sense it satisfies  $\int_0^T (Y_s - L_s) dK_s = 0$ . They proved existence and uniqueness of a solution when  $f$  is Lipschitz in  $(y, z)$  uniformly in  $(t, \omega)$ . Then Matoussi (1997, [10]) considered the case  $f$  continuous and at most linear growth in  $y, z$  and proved the existence of a maximal and a minimal solution.

Cvitanic and Karatzas (1995) studied the backward stochastic differential equation with two barriers. A solution to such equation associated to a terminal condition  $\xi$ , a coefficient  $f(t, \omega, y, z)$  and two barriers  $L$  and  $U$ , is a triple  $(Y, Z, K)$  of adapted processes, valued in

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$\mathbf{R}^{1+d+1}$ , which satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \text{ a.s.}$$

$L_t \leq Y_t \leq U_t$ ,  $0 \leq t \leq T$  and  $\int_0^T (Y_s - L_s) dK_s^+ = \int_0^T (Y_s - U_s) dK_s^- = 0$ , a.s. In this case, a solution  $Y$  has remain between the lower boundary  $L$  and upper boundary  $U$ , almost surely. This is achieved by the cumulative action of two continuous, increasing reflecting processes  $K^\pm$ , which act in a minimal way when  $Y$  attempts to cross barriers. In this paper, authors proves the existence and uniqueness of the solution, under certain condition of  $\xi$ ,  $L$  and  $U$ , and Lipschitz condition of generator  $g$ . In 1997, Hamadene, Lepeltier and Matoussi extended the existence result to the case when generator  $g$  is a continuous function.

In this paper we study the uniqueness of the solution of reflected BSDE with one or two barriers, under continuous and linear increasing condition of generator  $g$ . Before that we study the construction of solution of of reflected BSDE with one or two barriers.

The paper is organized as following: in following section, we present the basic assumption and notation of reflected BSDE with one or two barriers, then we give existence result of reflected BSDE with one or two barriers under linear increasing condition of  $g$  in section 3. In section 4 and 5, we study the construction of solution of of reflected BSDE with one or two barriers under linear increasing condition of  $g$ . In section 6, we give a sufficient result of uniqueness result of reflected BSDE with one or two barriers. In section 7, we consider continuous depending on  $c$  of coefficient  $g + c$ .

## 2 Assumptions and Notations

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space, and  $(B_t)_{0 \leq t \leq T} = (B_t^1, B_t^2, \dots, B_t^d)'_{0 \leq t \leq T}$  be a  $d$ -dimensional Brownian motion defined on a finite interval  $[0, T]$ ,  $0 < T < +\infty$ . Denote by  $\{\mathcal{F}_t; 0 \leq t \leq T\}$  the standard filtration generated by the Brownian motion  $B$ , i.e.  $\mathcal{F}_t$  is the completion of

$$\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\},$$

with respect to  $(\mathcal{F}, P)$ . We denote by  $\mathcal{P}$  the  $\sigma$ -algebra of predictable sets on  $[0, T] \times \Omega$ .

We shall need the following spaces:

$$\begin{aligned} \mathbf{L}^2(\mathcal{F}_t) &= \{\eta : \mathcal{F}_t\text{-measurable random real-valued variable, s.t. } E(|\eta|^2) < +\infty\}, \\ \mathcal{H}_n^2(0, T) &= \{(\psi_t)_{0 \leq t \leq T} : \text{predictable process valued in } \mathbb{R}^n, \text{ s.t. } E \int_0^T |\psi(t)|^2 dt < +\infty\}, \\ \mathcal{S}^2(0, T) &= \{(\psi_t)_{0 \leq t \leq T} : \text{progressively measurable real-valued continuous process,} \\ &\quad \text{s.t. } E(\sup_{0 \leq t \leq T} |\psi(t)|^2) < +\infty\}, \\ \mathcal{A}^2(0, T) &= \{(K_t)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T) : \text{increasing process, s.t. } K(0) = 0, E(K(T)^2) < +\infty\}. \end{aligned}$$

To simplify the symbol, we use  $\mathcal{H}^2(0, T)$  instead of  $\mathcal{H}_n^2(0, T)$ , when  $n = 1$ .

To consider a reflected BSDE with one or two barriers, we are first given **(i)** a terminal condition  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$ , then **(ii)** a function

$$g(t, \omega, y, z) : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R},$$

which is the coefficient of the reflected BSDE, and satisfies the followings:

- (H1) for  $(y, z) \in \mathbf{R} \times \mathbf{R}^d$ ,  $g(\cdot, t, y, z) \in \mathcal{H}^2(0, T)$ ;  
(H2) there exists a constant  $\beta > 0$ , such that for  $(t, \omega) \in [0, T] \times \Omega$ ,

$$|g(t, y, z)| \leq \beta(1 + |y| + |z|); \quad (1)$$

- (H3) for  $(t, \omega) \in [0, T] \times \Omega$ ,  $g(t, \omega, \cdot, \cdot)$  is continuous.

And we need **(iii)** barriers  $L$  and  $U$ , which are a progressively measurable real-valued continuous process, such that

$$E[\sup_{0 \leq t \leq T} (L_t^+)^2 + \sup_{0 \leq t \leq T} (U_t^-)^2] < \infty, \text{ and } L_T \leq \xi \leq U_T. \quad (2)$$

Moreover, we assume that there exists a semimartingale  $X$  with the form  $X_t = X_0 + \int_0^t \phi_s dB_s + V_t$ , where  $V$  is a process with finite variation with  $V = V^+ - V^-$  and  $V^\pm \in \mathcal{A}^2(0, T)$ , satisfies for  $t \in [0, T]$ ,

$$L_t \leq X_t \leq U_t.$$

**Definition 2.1.** We say a triple  $(y_t, z_t, K_t)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{A}^2(0, T)$  is a solution of reflected BSDE with one barrier associated to  $(\xi, g, L)$ , if it satisfies

- (i) for  $t \in [0, T]$ ,

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds + K_T - K_t - \int_t^T z_s dB_s;$$

- (ii)  $y_t \geq L_t$ , a.s.  $0 \leq t \leq T$ ;

- (iii)  $\int_0^T (y_t - L_t) dK_t = 0$ .

**Definition 2.2.** We say a quadruple  $(y_t, z_t, A_t, K_t)_{0 \leq t \leq T}$  is a solution of reflected BSDE with two barriers associated to  $(\xi, g, L, U)$ , for  $y \in \mathcal{S}^2(0, T)$ ,  $z \in \mathcal{H}_d^2(0, T)$  and  $K^\pm \in \mathcal{A}^2(0, T)$ , if it satisfies

- (i) for  $t \in [0, T]$ ,

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T z_s dB_s;$$

- (ii)  $L_t \leq y_t \leq U_t$ , a.s.  $0 \leq t \leq T$ ;

- (iii)  $\int_0^T (y_t - L_t) dK_t^+ = \int_0^T (y_t - U_t) dK_t^- = 0$ .

### 3 Preliminaries

In this section, we present some known results of reflected BSDE under linear increasing condition. For  $n \in \mathbf{N}$ , we define

$$\begin{aligned} \underline{g}_n(t, y, z) &= \inf_{u \in \mathbf{R}, v \in \mathbf{R}^d} \{g(t, u, v) + n(|y - u| + |z - v|)\}, \\ \overline{g}_n(t, y, z) &= \sup_{u \in \mathbf{R}, v \in \mathbf{R}^d} \{g(t, u, v) - n(|y - u| + |z - v|)\}. \end{aligned} \quad (3)$$

These sequences give important approximations of continuous functions by Lipschitz functions (see proof in [9]):

**Lemma 3.1.** *Let  $g$  satisfy (H1), (H2) and (H3). Set  $\mu = \max\{\beta, A\}$ , then for  $n > \mu$ , we have for  $t \in [0, T]$ ,  $y \in \mathbf{R}$ ,  $z \in \mathbf{R}^d$*

- (i)  $-\mu(|y| + |z| + 1) \leq \underline{g}_n(t, y, z) \leq g(t, y, z) \leq \bar{g}_n(t, y, z) \leq \mu(|y| + |z| + 1)$ ;
- (ii)  $\underline{g}_n(t, y, z)$  (resp.  $\bar{g}_n(t, y, z)$ ) is non-decreasing (resp. non-increasing) in  $n$ ;
- (iii)  $\underline{g}_n(t, y, z)$  and  $\bar{g}_n(t, y, z)$  are Lipschitz in  $(y, z)$  with parameter  $n$ , uniformly with respect to  $(t, \omega)$ ;
- (iv) If  $(y_n, z_n) \rightarrow (y, z)$ , as  $n \rightarrow \infty$ , then  $\underline{g}_n(t, y_n, z_n)$  (resp.  $\bar{g}_n(t, y_n, z_n)$ )  $\rightarrow g(t, y, z)$ , as  $n \rightarrow \infty$ .

For  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$  and  $L$  satisfying (iii), we consider two sequence of reflected BSDE associated to  $(\xi, \bar{g}_n, L)$  and  $(\xi, \underline{g}_n, L)$ , for  $n \in \mathbf{N}$ , respectively. For  $n > \mu$ ,  $\bar{g}_n$  and  $\underline{g}_n$  are Lipschitz in  $(y, z)$ , so reflected BSDEs have unique solutions in  $\mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{A}^2(0, T)$ , denoted as  $(\bar{y}^n, \bar{z}^n, \bar{K}^n)$  and  $(\underline{y}^n, \underline{z}^n, \underline{K}^n)$  respectively. Namely, the followings are satisfied:

$$\begin{aligned} \bar{y}_t^n &= \xi + \int_t^T \bar{g}_n(s, \bar{y}_s^n, \bar{z}_s^n) ds + \bar{K}_T^n - \bar{K}_t^n - \int_t^T \bar{z}_s^n dB_s, \\ \bar{y}_t^n &\geq L_t, \quad \int_0^T (\bar{y}_t^n - L_t) d\bar{K}_t^n = 0; \end{aligned} \quad (4)$$

and

$$\begin{aligned} \underline{y}_t^n &= \xi + \int_t^T \underline{g}_n(s, \underline{y}_s^n, \underline{z}_s^n) ds + \underline{K}_T^n - \underline{K}_t^n - \int_t^T \underline{z}_s^n dB_s, \\ \underline{y}_t^n &\geq L_t, \quad 0 \leq t \leq T, \quad \int_0^T (\underline{y}_t^n - L_t) d\underline{K}_t^n = 0. \end{aligned} \quad (5)$$

We can prove

**Lemma 3.2.** *Under assumption (i)-(iii), there exists a constant  $M_0$  independent of  $n$ , such that*

$$E\left[\sup_{0 \leq t \leq T} |\underline{y}_t^n|^2 + \int_0^T |\underline{z}_t^n|^2 dt + \sup_{0 \leq t \leq T} |\bar{y}_t^n|^2 + \int_0^T |\bar{z}_t^n|^2 dt\right] \leq M_0.$$

Then, we have the existence of the maximal and minimal solution.

**Theorem 3.1.** *Let  $(\xi, g, L)$  be a triple satisfying the above assumptions, in particular (i)-(iii). Then the reflected BSDE associated to  $(\xi, g, L)$ , has the maximal solution  $(\bar{y}_t, \bar{z}_t, \bar{K}_t)_{0 \leq t \leq T}$  (resp. minimal solution  $(\underline{y}_t, \underline{z}_t, \underline{K}_t)_{0 \leq t \leq T}$ ), i.e. it satisfies definition 2.1. Moreover  $t \in [0, T]$ ,*

$$\underline{y}_t^n \leq \underline{y}_t^{n+1} \leq \underline{y}_t \leq \bar{y}_t \leq \bar{y}_t^{n+1} \leq \bar{y}_t.$$

And  $(\bar{y}^n, \bar{z}^n, \bar{K}^n) \rightarrow (\bar{y}, \bar{z}, \bar{K})$  and  $(\underline{y}^n, \underline{z}^n, \underline{K}^n) \rightarrow (\underline{y}, \underline{z}, \underline{K})$  both in  $\mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{S}^2(0, T)$  as  $n \rightarrow \infty$ .

**Remark 3.1.** *In [10], theorem 1 give the proof of the existence of the minimal solution by the convergence of  $(\underline{y}^n, \underline{z}^n, \underline{K}^n)$ . Due to the maximal solution, we can prove its existence by considering  $(\bar{y}^n, \bar{z}^n, \bar{K}^n)$  symmetrically.*

**Remark 3.2.** We say the solution  $(\bar{y}_t, \bar{z}_t, \bar{K}_t)_{0 \leq t \leq T}$  (resp.  $(\underline{y}_t, \underline{z}_t, \underline{K}_t)_{0 \leq t \leq T}$ ) is maximal (resp. minimal) in the sense that, if there exists another triple  $(y'_t, z'_t, K'_t)_{0 \leq t \leq T}$  satisfies definition ??, then  $y'_t \leq \bar{y}_t$  (resp.  $\underline{y}_t \leq y'_t$ )  $0 \leq t \leq T$ . This is an easy result of the following comparison theorem.

We have the following comparison result:

**Theorem 3.2.** For  $i = 1, 2$ , assume  $(\xi^i, g^i, L^i)$  satisfying the assumptions (i)-(iii). Let  $(\bar{y}_t^i, \bar{z}_t^i, \bar{K}_t^i)_{0 \leq t \leq T}$  be the maximal solution of reflected BSDE associated to  $(\xi^i, g^i, L^i)$ . Moreover if for  $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$ , we have

$$\xi^1 \geq \xi^2, g^1(t, y, z) \geq g^2(t, y, z), L_t^1 \geq L_t^2,$$

then  $\bar{y}_t^1 \geq \bar{y}_t^2$ ,  $t \in [0, T]$ . Furthermore, if  $L^1 = L^2$ , then  $\bar{K}_t^1 - \bar{K}_s^1 \leq \bar{K}_t^2 - \bar{K}_s^2$ ,  $0 \leq s \leq t \leq T$ .

**Remark 3.3.** The comparison results still hold when we consider the minimal solutions  $(\underline{y}_t^i, \underline{z}_t^i, \underline{K}_t^i)_{0 \leq t \leq T}$  of corresponding reflected BSDEs.

**Remark 3.4.** The proof of this theorem can be found in [7], which follows from the approximation in the proof of theorem 3.1 and the comparison theorem for Lipschitz case in [1] and [6].

Then we recall a continuous dependence result of reflected BSDE under Lipschitz condition, which is proved in [1].

**Theorem 3.3.** Let  $(\xi^i, g^i, L)$  be two triple satisfying the assumptions (i), (iii) and  $g^i$  is Lipschitz in  $(y, z)$  uniformly in  $(t, \omega)$ , i.e. for some  $k > 0$ ,  $t \in [0, T]$ ,  $y, y' \in \mathbf{R}$ ,  $z, z' \in \mathbf{R}^d$ , such that

$$|g^i(t, y, z) - g^i(t, y', z')| \leq k(|y - y'| + |z - z'|).$$

Suppose  $(\bar{y}_t^i, \bar{z}_t^i, \bar{K}_t^i)_{0 \leq t \leq T}$  is the solution of reflected BSDE  $(\xi^i, g^i, L^i)$ , for  $i = 1, 2$ . Set

$$\begin{aligned} \Delta \xi &= \xi^1 - \xi^2, \Delta g = g^1 - g^2, \\ \Delta y &= y^1 - y^2, \Delta z = z^1 - z^2, \Delta K = K^1 - K^2. \end{aligned}$$

Then there exists a constant  $C$  such that

$$E\left[\sup_{0 \leq t \leq T} |\Delta y_t|^2 + \int_0^T |\Delta z_s|^2 ds + |\Delta K_T|^2\right] \leq CE[|\Delta \xi|^2 + \int_0^T |\Delta g(s, y_s^2, z_s^2)| ds].$$

For reflected BSDE with two continuous barriers, we have similar results of existence of maximin and minimum solution and comparison theorem.

**Theorem 3.4.** Consider reflected BSDE with two barriers associated to  $(\xi, g, L, U)$ , which satisfies assumptions (i)-(iii), it has the maximal solution  $(\bar{y}_t, \bar{z}_t, \bar{K}_t^+, \bar{K}_t^-)_{0 \leq t \leq T}$  (resp. minimal solution  $(\underline{y}_t, \underline{z}_t, \underline{K}_t^+, \underline{K}_t^-)_{0 \leq t \leq T}$ ), i.e. it satisfies definition 2.1. Moreover  $t \in [0, T]$ ,

$$\underline{y}_t^n \leq \underline{y}_t^{n+1} \leq \underline{y}_t \leq \bar{y}_t \leq \bar{y}_t^{n+1} \leq \bar{y}_t.$$

And  $(\bar{y}^n, \bar{z}^n, \bar{K}^{n+}, \bar{K}^{n-}) \rightarrow (\bar{y}, \bar{z}, \bar{K}^+, \bar{K}^-)$  and  $(\underline{y}^n, \underline{z}^n, \underline{K}^{n+}, \underline{K}^{n-}) \rightarrow (\underline{y}, \underline{z}, \underline{K}^+, \underline{K}^-)$  both in  $\mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{S}^2(0, T)$  as  $n \rightarrow \infty$ . Here  $(\bar{y}^n, \bar{z}^n, \bar{K}^{n+}, \bar{K}^{n-})$  (resp.  $(\underline{y}^n, \underline{z}^n, \underline{K}^{n+}, \underline{K}^{n-})$ ) is solution of reflected BSDE with two barriers associated to  $(\xi, \bar{g}_n, L, U)$  (resp.  $(\xi, \underline{g}_n, L, U)$ ).

**Theorem 3.5.** For  $i = 1, 2$ , assume  $(\xi^i, g^i, L^i, U^i)$  satisfying the assumptions (i)-(iii). Let  $(\bar{y}_t^i, \bar{z}_t^i, \bar{K}_t^{i+}, \bar{K}_t^{i-})_{0 \leq t \leq T}$  be the maximal solution of reflected BSDE associated to  $(\xi^i, g^i, L^i, U^i)$ . Moreover if for  $(t, y, z) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d$ , we have

$$\xi^1 \geq \xi^2, g^1(t, y, z) \geq g^2(t, y, z), L_t^1 \geq L_t^2, U_t^1 \geq U_t^2,$$

then  $\bar{y}_t^1 \geq \bar{y}_t^2$ ,  $t \in [0, T]$ . Furthermore, if  $L^1 = L^2$ ,  $U^1 = U^2$ , then  $\bar{K}_t^{1+} - \bar{K}_s^{1+} \leq \bar{K}_t^{2+} - \bar{K}_s^{2+}$ ,  $\bar{K}_t^{1-} - \bar{K}_s^{1-} \geq \bar{K}_t^{2-} - \bar{K}_s^{2-}$ ,  $0 \leq s \leq t \leq T$ .

**Remark 3.5.** The comparison results still hold when we consider the minimal solutions  $(\underline{y}_t^i, \underline{z}_t^i, \underline{K}_t^{i+}, \underline{K}_t^{i-})_{0 \leq t \leq T}$  of corresponding reflected BSDEs.

## 4 The solutions between $\bar{y}_t$ and $\underline{y}_t$ for BSDE with one lower barrier

Our first result is that between the maximal solution  $\bar{y}_t$  and the minimal solution  $\underline{y}_t$ , we can construct as many solution  $(y_t, z_t, K_t)_{0 \leq t \leq T}$  as we want to satisfy the reflected BSDE $(\xi, g, L)$ , i.e. definition 2.1 is satisfied.

**Theorem 4.1.** Assume that (i)-(iii) hold for  $(\xi, g, L)$ . Let  $(\underline{y}_t, \underline{z}_t, \underline{K}_t)_{0 \leq t \leq T}$  and  $(\bar{y}_t, \bar{z}_t, \bar{K}_t)_{0 \leq t \leq T}$  be the minimal and maximal solution of reflected BSDE $(\xi, g, L)$ , respectively, i.e. definition 2.1 is satisfied for both triples. Then for any  $t_0 \in [0, T]$ , and  $\eta \in \mathbf{L}^2(\mathcal{F}_{t_0})$ , such that

$$\underline{y}_{t_0} \leq \eta \leq \bar{y}_{t_0}, \quad a.s.,$$

there exists at least one solution  $(y_t, z_t, K_t)_{0 \leq t \leq T}$  in  $\mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{A}^2(0, T)$  of reflected BSDE $(\xi, g, L)$  passing through  $(t_0, \eta)$ , namely  $y_{t_0} = \eta$ .

**Proof.** On the interval  $[0, t_0]$ , since  $\eta \in \mathbf{L}^2(\mathcal{F}_{t_0})$ , there exists at least one triple  $(y_t^1, z_t^1, K_t^1)_{0 \leq t \leq t_0}$  to be a solution of reflected BSDE $(\eta, g, L)$  on  $[0, t_0]$ , i.e.

$$\begin{aligned} y_t^1 &= \eta + \int_t^{t_0} g(s, y_s^1, z_s^1) ds + K_{t_0}^1 - K_t^1 - \int_t^{t_0} z_s^1 dB_s, \\ y_t^1 &\geq L_t, 0 \leq t \leq t_0, \quad \int_0^{t_0} (y_t^1 - L_t) dK_t^1 = 0. \end{aligned}$$

Then we fix a process  $z^2 \in \mathcal{H}_d^2(t_0, T)$ , and consider a (strong) solution  $(y_t^2)_{t_0 \leq t \leq T}$  of the following SDE

$$y_t^2 = \eta - \int_{t_0}^t g(s, y_s^2, z_s^2) ds + \int_{t_0}^t z_s^2 dB_s.$$

Define a stopping time  $\tau = \inf\{t \geq t_0, y_t^2 \notin (\underline{y}_t, \bar{y}_t)\}$ . Since  $\underline{y}_T = \bar{y}_T$ , we get  $\tau < T$ . Notice that  $y_{t_0}^2 = \eta \geq L_{t_0}$ , and by the continuity of solutions we know that on the interval  $[t_0, \tau]$ ,  $y_t^2 \geq \underline{y}_t \geq L_t$ , which implies that on this interval  $(y_t^2, z_t^2, 0)_{t_0 \leq t \leq \tau}$  is also a solution to reflected BSDE $(\xi, g, L)$  on  $[t_0, \tau]$ .

Now we denote the triple on  $[0, T]$

$$\begin{aligned} y_t &= y_t^1 1_{[0, t_0)}(t) + y_t^2 1_{[t_0, \tau)}(t) + \bar{y}_t 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{y}_t 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}, \\ z_t &= z_t^1 1_{[0, t_0)}(t) + z_t^2 1_{[t_0, \tau)}(t) + \bar{z}_t 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{z}_t 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}, \\ K_t &= K_t^1 1_{[0, t_0)}(t) + K_t^1 1_{[t_0, \tau)}(t) + \bar{K}_t 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{K}_t 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}. \end{aligned}$$

It is easy to check that  $(y_t, z_t, K_t)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times \mathcal{A}^2(0, T)$  satisfies definition 2.1, which means that it is a solution of reflected BSDE associated to  $(\xi, g, L)$ .  $\square$

**Remark 4.1.** *This result is still true if we replace the linear increasing property of  $g$  (i.e. (H2)) by quadratic growth assumption of  $[\gamma]$ .*

## 5 Similar result for reflected BSDE with two barriers

We have similar result for reflected BSDE with two barriers for constructing as many solution  $(y_t, z_t, K_t^+, K_t^-)_{0 \leq t \leq T}$  as we want to satisfy the reflected BSDE $(\xi, g, L, U)$ , i.e. definition 2.2 is satisfied.

**Theorem 5.1.** *Assume that (i)-(iii) hold for  $(\xi, g, L, U)$ . Let  $(\underline{y}_t, \underline{z}_t, \underline{K}_t^+, \underline{K}_t^-)_{0 \leq t \leq T}$  and  $(\bar{y}_t, \bar{z}_t, \bar{K}_t^+, \bar{K}_t^-)_{0 \leq t \leq T}$  be the minimal and maximal solution of reflected BSDE $(\xi, g, L, U)$ , respectively, i.e. definition 2.2 is satisfied for both triples. Then for any  $t_0 \in [0, T]$ , and  $\eta \in \mathbf{L}^2(\mathcal{F}_{t_0})$ , such that*

$$\underline{y}_{t_0} \leq \eta \leq \bar{y}_{t_0}, \quad \text{a.s.},$$

*there exists at least one solution  $(y_t, z_t, K_t^+, K_t^-)_{0 \leq t \leq T}$  in  $\mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times (\mathcal{A}^2(0, T))^2$  of reflected BSDE $(\xi, g, L, U)$  passing through  $(t_0, \eta)$ , namely  $y_{t_0} = \eta$ .*

**Proof.** On the interval  $[0, t_0]$ , since  $\eta \in \mathbf{L}^2(\mathcal{F}_{t_0})$ , there exists at least one triple  $(y_t^1, z_t^1, K_t^{1+}, K_t^{1-})_{0 \leq t \leq t_0}$  to be a solution of reflected BSDE $(\eta, g, L, U)$  on  $[0, t_0]$ , i.e.

$$\begin{aligned} y_t^1 &= \eta + \int_t^{t_0} g(s, y_s^1, z_s^1) ds + K_{t_0}^{1+} - K_t^{1+} - (K_{t_0}^{1-} - K_t^{1-}) - \int_t^{t_0} z_s^1 dB_s, \\ L_t &\leq y_t^1 \leq U_t, 0 \leq t \leq t_0, \quad \int_0^{t_0} (y_t^1 - L_t) dK_t^{1+} = \int_0^{t_0} (y_t^1 - U_t) dK_t^{1-} = 0. \end{aligned}$$

Then we fix a process  $z^2 \in \mathcal{H}_d^2(t_0, T)$ , and consider a (strong) solution  $(y_t^2)_{t_0 \leq t \leq T}$  of the following SDE

$$y_t^2 = \eta - \int_{t_0}^t g(s, y_s^2, z_s^2) ds + \int_{t_0}^t z_s^2 dB_s.$$

Define a stopping time  $\tau = \inf\{t \geq t_0, y_t^2 \notin (\underline{y}_t, \bar{y}_t)\}$ . Since  $\underline{y}_T = \bar{y}_T$ , we get  $\tau < T$ . Notice that  $L_{t_0} \leq y_{t_0}^2 = \eta \leq U_{t_0}$ , and by the continuity of solutions we know that on the interval  $[t_0, \tau]$ ,  $L_t \leq \underline{y}_t \leq y_t^2 \leq \bar{y}_t \leq U_t$ , which implies that on this interval  $(y_t^2, z_t^2, 0, 0)_{t_0 \leq t \leq \tau}$  is also a solution to reflected BSDE $(\xi, g, L, U)$  on  $[t_0, \tau]$ .

Now we denote the triple on  $[0, T]$

$$\begin{aligned} y_t &= y_t^1 1_{[0, t_0)}(t) + y_t^2 1_{[t_0, \tau)}(t) + \bar{y}_t 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{y}_t 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}, \\ z_t &= z_t^1 1_{[0, t_0)}(t) + z_t^2 1_{[t_0, \tau)}(t) + \bar{z}_t 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{z}_t 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}, \\ K_t^+ &= K_t^{1+} 1_{[0, t_0)}(t) + K_{t_0}^{1+} 1_{[t_0, \tau)}(t) + \bar{K}_t^+ 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{K}_t^+ 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}}, \\ K_t^- &= K_t^{1-} 1_{[0, t_0)}(t) + K_{t_0}^{1-} 1_{[t_0, \tau)}(t) + \bar{K}_t^- 1_{[\tau, T]}(t) 1_{\{y_\tau = \bar{y}_\tau\}} + \underline{K}_t^- 1_{[\tau, T]}(t) 1_{\{y_\tau < \bar{y}_\tau\}} \end{aligned}$$

It is easy to check that  $(y_t, z_t, K_t^+, K_t^-)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T) \times \mathcal{H}_d^2(0, T) \times (\mathcal{A}^2(0, T))^2$  satisfies definition 2.2, which means that it is a solution of reflected BSDE associated to  $(\xi, g, L, U)$ .  $\square$

**Remark 5.1.** *This result is still true if we replace the linear increasing property of  $g$  (i.e. (H2)) by quadratic growth assumption as in [7].*

## 6 One uniqueness theorem when $g$ only depend on $z$

In this section, we will prove one uniqueness result of the reflected BSDE whose coefficient is only depend on  $z$ . We still assume (H1), (H2) and (H3) hold for  $g : [0, T] \times \Omega \times \mathbf{R}^d \rightarrow \mathbf{R}$ , however in order to get the uniqueness of the solution, we need

(H4) uniform continuity:  $g(t, \cdot)$  is uniformly continuous in  $z$ , uniformly with respect to  $(\omega, t)$ . More precisely, there exists a continuous, non-decreasing function  $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with linear growth with parameter  $A$  and satisfying  $\phi(0) = 0$  such that for any  $t \in [0, T]$ ,  $z_1, z_2 \in \mathbf{R}^d$ ,

$$|g(t, z_1) - g(t, z_2)| \leq \phi(|z_1 - z_2|).$$

In fact, this assumption implies assumption (H3).

Then as (3) we define

$$\begin{aligned} \underline{g}_n(z) &= \inf_{u \in \mathbf{R}^d} \{g(u) + n|z - u|\}, \\ \bar{g}_n(z) &= \sup_{u \in \mathbf{R}^d} \{g(u) - n|z - u|\}. \end{aligned}$$

So for  $n > \mu$ , Lemma 3.1 also holds for  $\underline{g}_n(z)$  and  $\bar{g}_n(z)$ . Furthermore we have

**Lemma 6.1.** *For  $z \in \mathbf{R}^d$ ,  $0 \leq g(z) - \underline{g}_n(z) \leq \phi(\frac{\mu}{n-\mu})$  and  $0 \leq \bar{g}_n(z) - g(z) \leq \phi(\frac{\mu}{n-\mu})$ .*

This lemma is proved in [3].

We first consider reflected BSDE with one barrier. For  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$  and  $L$  satisfying (iii), we consider two sequence of reflected BSDE associated to  $(\xi, \bar{g}_n, L)$  and  $(\xi, \underline{g}_n, L)$ , for  $n \in \mathbf{N}$ , respectively. For  $n > \mu$ ,  $\bar{g}_n$  and  $\underline{g}_n$  are Lipschitz in  $z$ , so reflected BSDEs have unique solutions, which are denoted as  $(\bar{y}^n, \bar{z}^n, \bar{K}^n)$  and  $(\underline{y}^n, \underline{z}^n, \underline{K}^n)$  respectively. So the followings are satisfied:

$$\begin{aligned} \bar{y}_t^n &= \xi + \int_t^T \bar{g}_n(s, \bar{z}_s^n) ds + \bar{K}_T^n - \bar{K}_t^n - \int_t^T \bar{z}_s^n dB_s, \\ \bar{y}_t^n &\geq L_t, \quad \int_0^T (\bar{y}_t^n - L_t) d\bar{K}_t^n = 0; \end{aligned} \tag{6}$$



and

$$\begin{aligned}\underline{y}_t^n &= \xi + \int_t^T \underline{g}_n(s, \underline{z}_s^n) ds + \underline{K}_T^n - \underline{K}_t^n - \int_t^T \underline{z}_s^n dB_s, \\ \underline{y}_t^n &\geq L_t, 0 \leq t \leq T, \quad \int_0^T (\underline{y}_t^n - L_t) d\underline{K}_t^n = 0.\end{aligned}\tag{7}$$

Similarly to [3], we have the following lemma.

**Lemma 6.2.** *Under assumption (H1), (H2) and (H4), let  $(\underline{y}_t, \underline{z}_t, \underline{K}_t)_{0 \leq t \leq T}$  (resp.  $(\overline{y}_t, \overline{z}_t, \overline{K}_t)_{0 \leq t \leq T}$ ) be the minimal (resp. maximal) solution of reflected BSDE $(\xi, g, L)$ , i.e. they satisfy definition 2.1. Then we have for  $n > \mu$ ,  $E[\overline{y}_t^n - \underline{y}_t^n] \leq 2\phi(\frac{\mu}{n-\mu})T$ .*

**Proof.** From (6) and (7), we get

$$\begin{aligned}\overline{y}_t^n - \underline{y}_t^n &= \int_t^T (\overline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n)) ds - \int_t^T (\overline{z}_s^n - \underline{z}_s^n) dB_s \\ &\quad + (\overline{K}_T^n - \overline{K}_t^n) - (\underline{K}_T^n - \underline{K}_t^n).\end{aligned}$$

Since  $\overline{g}_n(t, z) \geq \underline{g}_n(t, z)$ , for  $(t, z) \in [0, T] \times \mathbf{R}^d$ , by the comparison theorem in [6] or Theorem 3.2, we have

$$0 \leq \overline{K}_T^n - \overline{K}_t^n \leq \underline{K}_T^n - \underline{K}_t^n.$$

Then

$$0 \leq \overline{y}_t^n - \underline{y}_t^n \leq \widetilde{y}_t^n,\tag{8}$$

where

$$\widetilde{y}_t^n := \int_t^T (\overline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n)) ds - \int_t^T (\overline{z}_s^n - \underline{z}_s^n) dB_s.\tag{9}$$

Set  $\widehat{z}_t^n := \overline{z}_t^n - \underline{z}_t^n$  and

$$l_t^{n,j} = \begin{cases} \frac{\underline{g}_n(t, \widehat{z}_t^{n,j-1}) - \underline{g}_n(t, \widehat{z}_t^{n,j})}{\overline{z}_t^{n,j} - \underline{z}_t^{n,j}}, & \text{if } \overline{z}_t^{n,j} \neq \underline{z}_t^{n,j} \\ 0, & \text{if } \overline{z}_t^{n,j} = \underline{z}_t^{n,j} \end{cases},\tag{10}$$

where  $\widehat{z}_t^{n,j}$  is the vector whose first  $j$  components are equal to those of  $\underline{z}_t^n$  and whose of last  $d - j$  components are equal to those of  $\overline{z}_t^n$ , i.e.  $\widehat{z}_t^j = (\underline{z}_t^{n,1}, \dots, \underline{z}_t^{n,j}, \overline{z}_t^{n,j+1}, \dots, \overline{z}_t^{n,d})$ . Here  $l_t^j$  is  $j$ th vector of  $l$ . Note that

$$\begin{aligned}\overline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n) &= \overline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \overline{z}_s^n) + \underline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n) \\ &= \widetilde{g}_s^n + \underline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n),\end{aligned}$$

where  $\widetilde{g}_s^n := \overline{g}_n(s, \overline{z}_s^n) - \underline{g}_n(s, \overline{z}_s^n)$ , we know that  $(\widetilde{y}_t^n, \widehat{z}_t^n)$  satisfies the following linear BSDE on  $[0, T]$ ,

$$\widetilde{y}_t^n = \int_t^T (l_s^n \widehat{z}_s^n + \widetilde{g}_s^n) ds - \int_t^T \widehat{z}_s^n dB_s.$$

Since now  $\underline{g}_n$  is Lipschitz in  $z$  with parameter  $n$ ,  $|l_s^n| \leq n$ . Consider the solution of linear SDE  $q_t^n := \exp[\int_0^t l_s^n dB_s - \frac{1}{2} \int_0^t |l_s^n|^2 ds]$ , applying Itô's formula to  $q_t^n \tilde{y}_t^n$  on  $[t, T]$  and taking conditional expectation, then we get

$$\begin{aligned}\tilde{y}_t^n &= (q_t^n)^{-1} E[\int_t^T q_s^n \tilde{g}_s^n ds | \mathcal{F}_t] \\ &= E[\int_t^T (\int_t^s l_s^n dB_s - \frac{1}{2} \int_t^s |l_s^n|^2 ds) \tilde{g}_s^n ds | \mathcal{F}_t].\end{aligned}$$

By lemma 6.1, we have  $0 \leq \tilde{g}_s^n \leq 2\phi(\frac{\mu}{n-\mu})$ , for  $s \in [0, T]$ ,  $n > \mu$ . Therefore

$$\begin{aligned}E[\tilde{y}_t^n] &= E[\int_t^T (\int_t^s l_s^n dB_s - \frac{1}{2} \int_t^s |l_s^n|^2 ds) \tilde{g}_s^n ds] \\ &\leq 2\phi(\frac{\mu}{n-\mu}) E[\int_t^T (\int_t^s l_s^n dB_s - \frac{1}{2} \int_t^s |l_s^n|^2 ds) ds] \\ &\leq 2\phi(\frac{\mu}{n-\mu}) T,\end{aligned} \tag{11}$$

in view of  $E[\int_t^s l_s^n dB_s - \frac{1}{2} \int_t^s |l_s^n|^2 ds] = 1$ , for  $t \leq s \leq T$ , which follows from the fact that  $q^n$  is a exponential martingale. The result follows from (8) and (11).  $\square$

With these preparations, we present our main result of this section.

**Theorem 6.1.** *Assume assumptions (i), (iii) hold for  $\xi$  and  $L$ , and  $g : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}$  satisfies (H1), (H2) and (H4). The solution of reflected BSDE( $\xi, g, L$ ) is unique.*

**Proof.** From Lemma 6.2, we have  $E[\bar{y}_t^n - \underline{y}_t^n] \leq 2\phi(\frac{\mu}{n-\mu})T$ , for  $n > \mu$ . So  $E[\bar{y}_t^n - \underline{y}_t^n] \rightarrow 0$ , as  $n \rightarrow \infty$ . While Lemma 3.2 implies  $(\bar{y}_t^n - \underline{y}_t^n)$  is bounded in  $\mathcal{S}^2(0, T)$  uniformly in  $n$ , we get  $E[(\bar{y}_t^n - \underline{y}_t^n)^2] \rightarrow 0$ , as  $n \rightarrow \infty$ , in view of  $\bar{y}_t^n - \underline{y}_t^n \geq 0$ .

Let  $(\underline{y}_t, \underline{z}_t, \underline{K}_t)_{0 \leq t \leq T}$  (resp.  $(\bar{y}_t, \bar{z}_t, \bar{K}_t)_{0 \leq t \leq T}$ ) be the minimal (resp. maximal) solution of reflected BSDE( $\xi, g, L$ ), by the convergence result of Theorem 3.1, we obtain

$$E[(\bar{y}_t - \underline{y}_t)^2] \leq E[(\bar{y}_t^n - \bar{y}_t^n)^2] + E[(\bar{y}_t^n - \underline{y}_t^n)^2] + E[(\underline{y}_t^n - \underline{y}_t)^2] \rightarrow 0,$$

as  $n \rightarrow \infty$ . The proof is complete.  $\square$

For reflected BSDE with two barriers, we have similar result.

**Theorem 6.2.** *Assume assumptions (i), (iii) hold for  $\xi$ ,  $L$  and  $U$  and  $g : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}$  satisfies (H1), (H2) and (H4). The solution of reflected BSDE( $\xi, g, L, U$ ) is unique.*

**Proof.** The proof is similar to the one of Theorem 6.1. First we need to prove a result as Lemma 6.2: for  $n > \mu$ ,  $E[\bar{y}_t^n - \underline{y}_t^n] \leq 2\phi(\frac{\mu}{n-\mu})T$ , where  $\bar{y}_t^n$  (resp.  $\underline{y}_t^n$ ) is the solution of reflected BSDE( $\xi, \bar{g}^n, L, U$ ) (resp.  $(\xi, \underline{g}^n, L, U)$ ). In fact, when we consider the difference of  $\bar{y}_t^n$  and  $\underline{y}_t^n$ .

$$\begin{aligned}\bar{y}_t^n - \underline{y}_t^n &= \int_t^T (\bar{g}_n(s, \bar{z}_s^n) - \underline{g}_n(s, \underline{z}_s^n)) ds - \int_t^T (\bar{z}_s^n - \underline{z}_s^n) dB_s \\ &\quad + (\bar{K}_T^{n+} - \bar{K}_t^{n+}) - (\underline{K}_T^{n+} - \underline{K}_t^{n+}) - (\bar{K}_T^{n-} - \bar{K}_t^{n-}) + (\underline{K}_T^{n-} - \underline{K}_t^{n-}).\end{aligned}$$

Thanks to Theorem 3.5, we have  $0 \leq \overline{K}_T^{n+} - \overline{K}_t^{n+} \leq \underline{K}_T^{n+} - \underline{K}_t^{n+}$ ,  $\overline{K}_T^{n-} - \overline{K}_t^{n-} \geq \underline{K}_T^{n-} - \underline{K}_t^{n-} \geq 0$ . Then we get  $0 \leq \overline{y}_t^n - \underline{y}_t^n \leq \widetilde{y}_t^n$ , where  $\widetilde{y}_t^n$  is the solution of (9). So following same method, it follows for  $n > \mu$ ,  $E[\overline{y}_t^n - \underline{y}_t^n] \leq 2\phi(\frac{\mu}{n-\mu})T$ .

Let  $(\underline{y}_t, \underline{z}_t, \underline{K}_t^+, \underline{K}_t^-)_{0 \leq t \leq T}$  (resp.  $(\overline{y}_t, \overline{z}_t, \overline{K}_t^+, \overline{K}_t^-)_{0 \leq t \leq T}$ ) be the minimal (resp. maximal) solution of reflected BSDE( $\xi, g, L, U$ ), by the convergence result of Theorem 3.4, we obtain

$$E[(\overline{y}_t - \underline{y}_t)^2] \leq E[(\overline{y}_t - \overline{y}_t^n)^2] + E[(\overline{y}_t^n - \underline{y}_t^n)^2] + E[(\underline{y}_t^n - \underline{y}_t)^2] \rightarrow 0,$$

as  $n \rightarrow \infty$ . The proof is complete.  $\square$

## 7 Uniqueness results about the Disturbance of Coefficient

In this section, we consider the uniqueness problem of solution for reflected BSDEs with respect to a disturbance  $c \in \mathbf{R}$  of its coefficient as  $g(t, y, z) + c$ , under the uniform continuity assumption of  $g$  on  $(y, z)$ . More precisely, we assume that the function  $g : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}$ , satisfies

(H4') uniform continuity:  $g(t, \cdot, \cdot)$  is uniformly continuous in  $(y, z)$ , uniformly with respect to  $(\omega, t)$ , i.e., there exists a continuous, non-decreasing function  $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  satisfying  $\phi(0) = 0$  and  $\phi(x) \leq A(1 + |x|)$  such that for any  $t \in [0, T]$ ,  $y_1, y_2 \in \mathbf{R}$ ,  $z_1, z_2 \in \mathbf{R}^d$ ,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \phi(|y_1 - y_2| + |z_1 - z_2|).$$

In fact, this assumption implies assumption (H3).

We define  $\underline{g}_n(t, y, z)$  and  $\overline{g}_n(t, y, z)$  same as in (3), then we know that lemma 3.1 holds for  $\underline{g}_n$  and  $\overline{g}_n$ . Moreover we have the following lemma, which is proved in [4]:

**Lemma 7.1.** *Let  $g$  satisfy (H1), (H2) and (H4'). For  $n \in \mathbf{N}$ , Set  $\mu = \max\{\beta, A\}$ , then for  $n > \mu$ , we have for  $t \in [0, T]$ ,  $y \in \mathbf{R}$ ,  $z \in \mathbf{R}^d$ ,  $0 \leq g(t, y, z) - \underline{g}_n(t, y, z) \leq \phi(\frac{\mu}{n-\mu})$  and  $0 \leq \overline{g}_n(t, y, z) - g(t, y, z) \leq \phi(\frac{\mu}{n-\mu})$ .*

Set  $(\underline{y}_t^c, \underline{z}_t^c, \underline{K}_t^c)_{0 \leq t \leq T}$  (resp.  $(\overline{y}_t^c, \overline{z}_t^c, \overline{K}_t^c)_{0 \leq t \leq T}$ ) be the minimal (resp. maximal) solution of reflected BSDE( $\xi, g + c, L$ ). Before considering the main result of this section, we prove a comparison theorem.

**Lemma 7.2.** *Assume (H1), (H2) and (H4') hold for  $g$ . For a given  $c > 0$ ,  $\xi \in \mathbf{L}^2(\mathcal{F}_T)$  and  $L$  satisfying (iii), set  $(y_t, z_t, K_t)_{0 \leq t \leq T}$  be a solution of reflected BSDE( $\xi, g, L$ ), then we have*

$$\underline{y}_t^c \geq y_t, t \in [0, T].$$

**Proof.** By lemma 7.1, (i) of lemma 3.1 and the continuity of  $\phi$  at  $x = 0$ , there exists an enough large  $n_0 > \mu$ , such that  $\phi(\frac{\mu}{n_0 - \mu}) < \frac{c}{2}$  and

$$g(t, y, z) \leq \underline{g}_{n_0}(t, y, z) + \phi(\frac{\mu}{n_0 - \mu}) \leq \underline{g}_{n_0}(t, y, z) + \frac{c}{2} < g(t, y, z) + c.$$

Notice that  $\underline{g}_{n_0}$  is Lipschitz in  $(y, z)$  with parameter  $n_0$ , so the reflected BSDE( $\xi, \underline{g}_{n_0} + \frac{c}{2}, L$ ) admits the unique solution defined as  $(y_t^{c, n_0}, z_t^{c, n_0}, K_t^{c, n_0})$ . By the comparison theorem 3.2, we get

$$y_t \leq \bar{y}_t \leq y_t^{c, n_0} \leq \underline{y}_t^c, t \in [0, T].$$

□

Now we introduce two auxiliary functions

$$\bar{m}(t, c) = E[\bar{y}_t^c], \underline{m}(t, c) = E[\underline{y}_t^c],$$

which have following properties:

**Proposition 7.1.** (i) For any  $c \in \mathbf{R}$ ,  $t \rightarrow \bar{m}(t, c)$  or  $\underline{m}(t, c)$  is continuous;  
(ii) for any  $t \in [0, T]$ ,  $c \rightarrow \bar{m}(t, c)$  or  $\underline{m}(t, c)$  is nondecreasing;  
(iii) for any  $t \in [0, T]$ ,  $c \rightarrow \bar{m}(t, c)$  is left continuous and  $c \rightarrow \underline{m}(t, c)$  is right continuous.

The proof is similar to the proposition 7 in [4], with the helps of comparison theorem 3.2, theorem 3.3 and the convergence results of theorem 3.1. So we omit here.

From the properties of  $\bar{m}(t, c)$  and  $\underline{m}(t, c)$  and lemma 7.2, we obtain a necessary and sufficient condition for the uniqueness of solution of reflected BSDE under uniformly continuous property.

**Theorem 7.1.** Let  $(\xi, g, L)$  satisfy (i)-(iii) and  $(H_4')$  hold for  $g$ . For  $c_0 \in \mathbf{R}$ , the following statements are equivalent:

- (i) The reflected BSDE( $\xi, g + c_0, L$ ) admits the unique solution;
- (ii)  $\underline{m}(t, c)$  is continuous at  $c = c_0$ , for all  $t \in [0, T]$ ;
- (iii)  $\bar{m}(t, c)$  is continuous at  $c = c_0$ , for all  $t \in [0, T]$ ;
- (iv)  $\bar{m}(t, c_0) = \bar{m}(t, c_0)$ , for all  $t \in [0, T]$ .

**Proof.** It is easy to check that (i) and (iv) are equivalent. Thanks to proposition 7.1 and comparison theorem 3.2, with the similar proof in theorem 9 in [4], we deduce that (i)  $\Rightarrow$  (ii) or (iii). Similarly, by lemma 7.2, we can prove (ii) or (iii)  $\Rightarrow$  (i). □

Finally, we give the last result of this section.

**Theorem 7.2.** Let  $(\xi, g, L)$  satisfy (i)-(iii) and  $(H_4')$  hold for  $g$ . Then the set of real number  $c$  such that the reflected BSDE with a disturbance  $c \in \mathbf{R}$  of its coefficient, i.e. reflected BSDE( $\xi, g + c, L$ ), admits more than one solution, is at most countable.

**Proof.** From theorem 7.1, we deduce that  $\bar{m}(t, c_0) = \bar{m}(t, c_0)$ , for all  $t \in [0, T]$ , is equivalent to the uniqueness of solution of reflected BSDE( $\xi, g + c, L$ ). Since  $c \rightarrow \bar{m}(t, c)$  or  $\underline{m}(t, c)$  is monotone, it has at most countable discontinuous points. While  $t \rightarrow \bar{m}(t, c)$  or  $\underline{m}(t, c)$  is continuous, our results follow by classical techniques. □

## References

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